Maximum Number of Transition Points in 3D Linear Symmetric Tensor Fields

Yue Zhang, Lawrence Roy, Ritesh Sharma, and Eugene Zhang

Abstract Transition points are well defined topological features in 3D tensor fields, which are important for the study of other prominent topological singularities such as wedges and trisectors. In this paper, we study the maximum number of transition points in a linear tensor field, which is important to process wedge and trisector classification along degenerate curves.

1 Introduction

3D symmetric tensor field topology consists of degenerate curves and neutral surfaces [6]. Transition points, which are degenerate points that separate wedge and trisector segments along a degenerate curve, are important topological features that are key to the study of 3D symmetric tensor field topology [10].

To the best of our knowledge, existing degenerate curve extraction methods do not explicitly extract transition points. We believe that this is largely due to the fact that it is not known how many transition points can exist in a tensor field and how to algebraically characterize them. In this paper, we attempt to address this cause by

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studying the minimum and maximum numbers of transition points in a 3D linear symmetric tensor field.

2 Previous Work

There has been much work on the analysis and visualization of 2D and 3D tensor fields. We refer the readers to the recent survey by Kratz et al. [5]. Here we only refer to the research most relevant to this chapter.

Delmarcelle and Hesselink [1, 2] introduce the topology of 2D symmetric tensor fields. They point out that there are two *fundamental* types of degenerate points in a 2D symmetric tensor field, i.e., wedges and trisectors, which have a tensor index of $\frac{1}{2}$ and $-\frac{1}{2}$, respectively. Hesselink et al. later extend this work to 3D symmetric tensor fields [4] and study triple degenerate points, i.e., all eigenvalues are the same. Zheng et al. [9] point out that triple degeneracies are not structurally stable features. They further show that double degeneracies, i.e., tensors with only two equal eigenvalues, form lines in the domain. In this work and subsequent research [11], they provide a number of degenerate curve extraction methods based on the analysis of the discriminant function of the tensor field. Furthermore, Zheng et al. [10] point out that near degenerate curves the tensor field exhibits 2D degenerate patterns and define separating surfaces which are extensions of separatrices from 2D symmetric tensor field topology. Zhang et al. [8] show that there are at least two and at most four degenerate curves in a 3D *linear* symmetric tensor field under structurally stable conditions. In this paper, we explore the minimum and maximum number of transition points in a 3D linear tensor field.

3 Background on Symmetric Tensors and Tensor Fields

We review some pertinent technical concepts in this section on tensors and tensor fields.

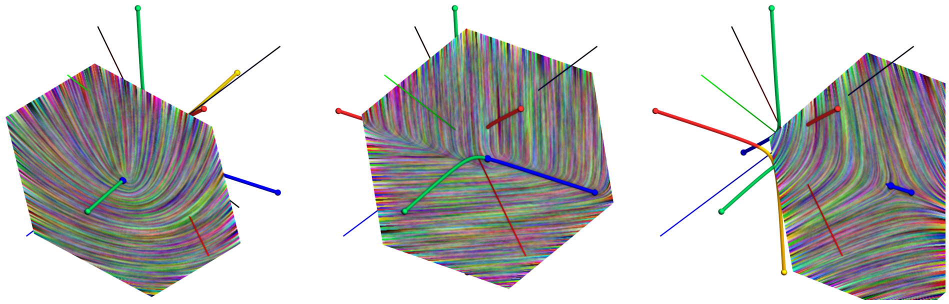
A K dimensional (symmetric) tensor T has K real-valued *eigenvalues*: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_K$. The largest and smallest eigenvalues are referred to as the *major eigenvalue* and *minor eigenvalue*, respectively. When $K = 3$, the middle eigenvalue is referred to as the *medium eigenvalue*. An eigenvector belonging to the major eigenvalue is referred to as a *major eigenvector*. Medium and minor eigenvectors can be defined similarly. Eigenvectors belonging to different eigenvalues are mutually perpendicular. A tensor is *degenerate* if there are repeating eigenvalues. In this case, there exists at least one eigenvalue whose corresponding eigenvectors form a higher-dimensional space than a line. When $K = 2$, a degenerate tensor must be a multiple of the identity matrix. When $K = 3$, there are two types of degenerate tensors, corresponding to three repeating eigenvalues (*triple degenerate*) and two repeating eigenvalues (*double degenerate*), respectively. There are two types of double degenerate tensors: (1) linear ($\lambda_1 > \lambda_2 = \lambda_3$) and (2) planar ($\lambda_1 = \lambda_2 > \lambda_3$). The trace of a tensor $T = (t_{ij})$ is $trace(T) = \sum_{i=1}^{3} \lambda_i$. T can be uniquely decomposed as $D + A$ where $D = \frac{trace(T)}{3}\mathbb{I}$ ($\mathbb{I}$ is the three-dimensional identity matrix) and $A = T - D$. The *deviator* A is a *traceless* tensor, i.e., $trace(A) = 0$. Note that T is degenerate if and only if A is degenerate. Consequently, it is sufficient to study the set of traceless tensors, which is closed under matrix addition and scalar multiplication.

A *tensor field* is a tensor-valued function over some domain $\Omega \subset \mathbb{R}^3$. The topology of a tensor field is defined as the set of *degenerate points*, i.e., points in the domain where the tensor field becomes degenerate.

In a 2D tensor field, there are two fundamental types of degenerate points, wedges and trisectors. They can be classified based on an invariant $\delta = \left| \begin{pmatrix} \frac{a_{11}-a_{22}}{2} & a_{12} \\ \frac{b_{11}-b_{22}}{2} & b_{12} \end{pmatrix} \right|$, where $a_{ij} = \frac{\partial t_{ij}(x,y)}{\partial x}$ and $b_{ij} = \frac{\partial t_{ij}(x,y)}{\partial y}$, i.e., the partial derivatives of the ij-th entry of the tensor field. A degenerate point $\mathbf{p}_0$ is a wedge when $\delta(\mathbf{p}_0) > 0$ and a trisector when $\delta(\mathbf{p}_0) < 0$. When $\delta(\mathbf{p}_0) = 0$, $\mathbf{p}_0$ is a higher-order degenerate point, which is structurally unstable.

In 3D symmetric tensor fields, a degenerate point can be classified by the linear-planar classification and the wedge-trisector classification. In the former, a degenerate point is either *triple degenerate*, *linear degenerate*, or *planar degenerate*. While triple degeneracies can exist, they are structurally unstable, i.e., they can disappear under arbitrarily small perturbations. In contrast, linear and planar degenerate points are structurally stable, i.e., they persist under sufficiently small perturbations in the tensor field. Moreover, under structurally stable conditions such points form curves, along which the tensor field is either always linear degenerate or always planar degenerate. While it is possible that linear and planar degenerate points are isolated points or form surfaces and volumes, these three scenarios do not persist under arbitrarily small perturbation in the field, i.e., are structurally unstable.

A degenerate point can also be classified based on the so-called wedge-trisector classification. Given a degenerate point $\mathbf{p}_0$, let $n = (\alpha, \beta, \gamma)$ be the non-repeating eigenvector at $\mathbf{p}_0$. The plane P that passes through $\mathbf{p}_0$ whose normal is n is referred to the *repeating plane* at $\mathbf{p}_0$. When projecting the 3D tensor field onto P, one obtains a 2D symmetric tensor field which, under structurally stable conditions, has exactly one degenerate point, $\mathbf{p}_0$. In the 2D tensor field, $\mathbf{p}_0$ can be either a wedge, a trisector, or a higher-order and thus structurally unstable degenerate point. In these cases, $\mathbf{p}_0$ will be referred to respectively as a wedge, a trisector, and a transition point in the 3D tensor field. Figure 1 demonstrates this with a 3D tensor field. Here and in the remaining figures in the paper, we use the following color scheme for degenerate points: yellow (linear wedge), green (planar wedge), red (linear trisector), and blue (planar trisector).



(a) (b) (c)

Fig. 1 Along a degenerate curve, the projection of the tensor field onto the repeating planes can exhibit 2D degenerate patterns such as a wedge (a) or a trisector (c). Between segments of wedges (green) and trisectors (blue), transition points can appear (b).

Note that while a higher-order degenerate point is structurally unstable, a transition point is structurally stable in 3D tensor fields. Moreover, a transition point is *not* the same as triple degenerate points. At the transition point, the repeating plane is tangent to the degenerate curve.

4 Transition Points in 3D Symmetric, Traceless Tensor Fields

We first note that the set of all traceless and symmetric tensors with configuration $\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & -a-d \end{pmatrix}$ form a five dimensional linear space $\mathbb{T}$ spanned by the basis $T_a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $T_d = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $T_b = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $T_c = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, and $T_e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Any tensor in this space can be expressed as $t_aT_a + t_bT_b + t_cT_c + t_dT_d + t_eT_e$ for some $t_a, t_b, t_c, t_d, t_e \in \mathbb{R}$. For convenience, it can be written in the vector form $(t_a, t_d, t_b, t_c, t_e)$.

A 3D symmetric, traceless linear tensor field has the following form

$$LT(x,y,z) = T_0 + xT_x + yT_y + zT_z \tag{1}$$

where $T_0 = \begin{pmatrix} a_0 & b_0 & c_0 \\ b_0 & d_0 & e_0 \\ c_0 & e_0 & -a_0-d_0 \end{pmatrix}$, $T_x = \begin{pmatrix} a_x & b_x & c_x \\ b_x & d_x & e_x \\ c_x & e_x & -a_x-d_x \end{pmatrix}$, $T_y = \begin{pmatrix} a_y & b_y & c_y \\ b_y & d_y & e_y \\ c_y & e_y & -a_y-d_y \end{pmatrix}$, and $T_z = \begin{pmatrix} a_z & b_z & c_z \\ b_z & d_z & e_z \\ c_z & e_z & -a_z-d_z \end{pmatrix}$ are symmetric, traceless matrices. Under structurally stable conditions, T_0, T_x, T_y, and T_z are linearly independent. In this section we study the number of transition points in such a tensor field.

Zhang et al. [8] shows that the degenerate points in a 3D linear tensor field satisfy the following system of equations

$$h(\alpha,\beta,\gamma) = 0 \tag{2}$$
$$\alpha^2+\beta^2+\gamma^2 = 1 \tag{3}$$

where (α, β, γ) is a unit non-repeating eigenvector and $h(\alpha,\beta,\gamma)$ is a homogeneous quadratic polynomial.

A transition point, as a degenerate point, must satisfy Equations 2 and 3. However, while degenerate points form curves under structurally stable conditions, transition points are isolated points. This indicates that one more condition is needed in terms of α, β, and γ.

Given a linear symmetric tensor field $LT(x,y,z) = T_0 + xT_x + yT_y + zT_z$, its projection onto any plane is also a 2D linear tensor field [7]. Consequently, the discriminant function δ is constant for the plane. We define a plane to be a *wedge plane* if $\delta > 0$, a *trisector plane* if $\delta < 0$, and more relevantly a *transition plane* if $\delta = 0$. A transition point must have its repeating plane as a transition plane. Therefore, characterizing transition planes gives us the additional condition to characterize transition points.

The following result from [7] is important to our analysis of transition planes.

Theorem 1. *Given a 3D linear tensor field $LT = T_0 + xT_x + yT_y + zT_z$ and a plane P, the discriminant function δ of the projection of LT onto P is a function of only T_x, T_y, and T_z.*

This leads to the following results:

Corollary 1. *Given a 3D linear tensor field $LT = T_0 + xT_x + yT_y + zT_z$ and two parallel planes P_1 and P_2, then P_1 is a transition plane if and only if P_2 is a transition plane.*

Corollary 2. *Given two 3D linear tensor field $LT = T_0 + xT_x + yT_y + zT_z$ and $LT' = T_0' + xT_x + yT_y + zT_z$, then a plane P is a transition plane for LT if and only if P is also a transition plane for LT'.*

Corollary 1 states that whether a plane is a transition plane depends only on the normal of the plane. Therefore, it is sufficient to only consider planes $P : \alpha x + \beta y + \gamma z = 0$, where $(\alpha, \beta, \gamma)^t$ is a unit vector and can be modelled by $\mathbb{RP}^2$, the two-dimensional real projective space.

Corollary 2 states that adding a constant tensor to the whole field will not change whether a plane is a transition plane. We can therefore set $T_0 = 0$ while finding the transition planes. Under these simplification conditions, we have the following result:

Lemma 1. *Given a linear symmetric tensor field $LT(x,y,z) = xT_x + yT_y + zT_z$, its projection onto the plane $P : \alpha x + \beta y + \gamma z = n \cdot p = 0$ has either one degenerate point or a line of degenerate points. The former occurs when $\delta \neq 0$ while the latter occurs when $\delta = 0$, i.e., transition plane.*

Proof. Select a coordinate system (O, X', Y') for the plane P where O is the origin. Then the projection tensor field has the following form in this coordinate system:

$$\begin{pmatrix} a_1x' + a_2y' & b_1x' + b_2y' \\ b_1x' + b_2y' & c_1x' + c_2y' \end{pmatrix} \tag{4}$$

Consequently, a degenerate point in T' satisfies:

$$a_1x' + a_2y' = c_1x' + c_2y' \tag{5}$$
$$b_1x' + b_2y' = 0 \tag{6}$$

The above system corresponds to the intersection of two lines in P. Either the two lines intersect at one point, i.e., the degenerate point, or they are the same line, i.e., every point on the line is a degenerate point. These two cases correspond precisely to the conditions $\delta \neq 0$ and $\delta = 0$, respectively. □

This characterization of transition planes is essential in our analysis of the maximum number of transition points.

We now consider $Q_n \subset \mathbb{T}$, the set of traceless, symmetric tensors whose projection onto the plane $P : n \cdot p = 0$ are 2D degenerate tensors (not necessarily traceless). Note that the set of 2D degenerate tensors is a codimension-two linear subspace of the set of 2D symmetric tensors. Therefore, since the projection is clearly surjective, Q_n is also a codimension-two linear subspace of $\mathbb{T}$. That is, Q_n is a three-dimensional linear subspace of $\mathbb{T}$. Q_n can be parameterized by a three-dimensional linear subspace W as follows.

Let q_n be the linear map from W which is isomorphic to $\mathbb{R}^3$ to the set of 3×3 symmetric matrices defined as

$$q_n(r) = \frac{1}{2}(n\,r^t + r\,n^t) - \frac{1}{3}(r \cdot n)\mathbb{I} \tag{7}$$

Note that $trace(q_n(r)) = trace(\frac{1}{2}(n\,r^t + r\,n^t) - \frac{1}{3}(r \cdot n)\mathbb{I}) = \frac{1}{2}(trace(n\,r^t) + trace(r\,n^t)) - \frac{1}{3}r \cdot ntrace(\mathbb{I}) = \frac{1}{2}(trace(r^t\,n) + trace(n^t\,t)) - r \cdot n = 0$. Consequently, q_n is a map from $\mathbb{R}^3$ to $\mathbb{T}$, the set of traceless tensors.

Furthermore, we show that $q_n(r) \in Q_n$ for any $r \in \mathbb{R}^3$. To see this, we return to the domain of the linear tensor field (which is not W) and consider the plane P_n. We can choose a coordinate system X', Y' for the plane. Let $M = \begin{pmatrix} X' & Y' \end{pmatrix}$, which is a 3×2 matrix. The projection of a 3×3 tensor K onto the plane P_n is therefore $M^t KM$. In particular, $M^t q_n(r)M = M^t(\frac{1}{2}(n\,r^t + r\,n^t) - \frac{1}{3}(r \cdot n)\mathbb{I})M = \frac{1}{2}M^t n\,r^t M + \frac{1}{2}M^t r\,n^t M - \frac{1}{3}(r \cdot n)M^t M$. Since X' and Y' are both perpendicular to n, we have $M^t n = n^t M = 0$. Furthermore, $M^t M$ is the 2D identity matrix, i.e., degenerate. Consequently, $q_n(r) \in Q_n$ for any $r \in \mathbb{R}^3$. This means q_n is a map from $\mathbb{R}^3$ to Q_n.

Furthermore, the map is an injection. This can be verified by studying the kernel of the map, i.e., for what $r = (u,v,w) \in W$, $q_n(r)$ is the zero tensor. Note that $q_n(u,v,w) = \begin{pmatrix} \frac{2\alpha u - \beta v - \gamma w}{3} & \frac{\beta u + \alpha v}{2} & \frac{\gamma u + \alpha w}{2} \\ \frac{\beta u + \alpha v}{2} & \frac{-\alpha u + 2\beta v - \gamma w}{3} & \frac{\beta w + \gamma v}{2} \\ \frac{\gamma u + \alpha w}{2} & \frac{\beta w + \gamma v}{2} & \frac{-\alpha u - \beta v + 2\gamma w}{3} \end{pmatrix}$. Given any non-zero $n = (\alpha, \beta, \gamma)$, the matrix $\begin{pmatrix} \frac{2\alpha}{3} & -\frac{\beta}{3} & -\frac{\gamma}{3} \\ -\frac{\alpha}{3} & \frac{2\beta}{3} & -\frac{\gamma}{3} \\ \frac{\beta}{2} & \frac{\alpha}{2} & 0 \\ \frac{\gamma}{2} & 0 & \frac{\alpha}{2} \\ 0 & \frac{\gamma}{2} & \frac{\beta}{2} \end{pmatrix}$ is rank 3. Therefore, for the following equations to hold,

$$\frac{2\alpha u - \beta v - \gamma w}{3} = 0 \tag{8}$$

$$\frac{-\alpha u - \beta v + 2\gamma w}{3} = 0 \tag{9}$$

$$\frac{\beta u + \alpha v}{2} = 0 \tag{10}$$

$$\frac{\gamma u + \alpha w}{2} = 0 \tag{11}$$

$$\frac{\beta w + \gamma v}{2} = 0 \tag{12}$$

we must have $u = v = w = 0$. That is, q_n is an injection. This means that q_n has rank 3, which is also the dimension of Q_n, so the map must be a surjection. Therefore, q_n is an isomorphism between W and Q_n, i.e., W is a parameterization of Q_n.

Thus far, we have identified a parameterization for Q_n, the set of symmetric, traceless tensors whose projection onto the plane P_n is degenerate. Given the tensor field $LT(x,y,z) = xT_x + yT_y + zT_z$, it maps $\mathbb{R}^3$ isomorphically to $U \subset \mathbb{T}$. When restricted on the plane P_n, LT maps P_n (isometric to $\mathbb{R}^2$) to a two-dimensional linear subspace $N \subset \mathbb{T}$. Under structurally stable conditions, $N_0 = N \bigcap Q_n$ is a zero-dimensional linear subspace, i.e., the zero tensor. This happens when P_n is *not* a transition plane. On the other hand, when N_0 is a one-dimensional linear space, i.e., there exists a point $p_0 = (x_0, y_0, z_0) \in P_n$ (p_0 is not the origin in $\mathbb{R}^3$) such that $x_0T_x + y_0T_y + z_0T_z \in Q_n$, then P_n is a transition plane.

In this case, $x_0T_x + y_0T_y + z_0T_z = q_n(r_0)$ for some non-zero $r_0 \in W$. Recall that U is a codimension-two subset of $\mathbb{T}$, so there exist two linear functions such that

$$f_0(q_n(r_0)) = 0 \tag{13}$$
$$g_0(q_n(r_0)) = 0 \tag{14}$$

Note that q_n is linear in terms of r_0. Consequently, both $f_0'(r_0) = f_0(q_n(r_0))$ and $g_0'(r_0) = g_0(q_n(r_0))$ are linear in terms of r_0.

Furthermore, p_0 is in P_n and so is perpendicular to n, i.e., $(x_0, y_0, z_0) \cdot n = 0$. Because T_x, T_y, and T_z are linearly independent, the linear map $LT\colon \mathbb{R}^3 \to \mathbb{T}$ given by the field will have a left inverse $TL\colon \mathbb{T} \to \mathbb{R}^3$, such that $TL(LT(\mathbf{p})) = \mathbf{p}$, for any $\mathbf{p} \in \mathbb{R}^3$ (e.g. TL could be LT's pseudoinverse). Since $q_n(r_0) = LT(p_0)$, $TL(q_n(r_0)) = p_0$. Therefore, we have

$$0 = n \cdot p_0 = n \cdot TL(q_n(r_0)) \tag{15}$$

Let $d_0(r_0) = n \cdot TL(q_n(r_0))$. This function is also a linear function of r_0, since both TL and q_n are linear functions with respect to their arguments. Therefore, r_0 must satisfy the following system of linear equations:

$$f_0'(r_0) = 0 \tag{16}$$
$$g_0'(r_0) = 0 \tag{17}$$
$$d_0(r_0) = 0 \tag{18}$$

The above system can be rewritten as

$$y_f \cdot r_0 = 0 \tag{19}$$
$$y_g \cdot r_0 = 0 \tag{20}$$
$$y_d \cdot r_0 = 0 \tag{21}$$

where y_f and y_g are vector-valued linear functions of n. To understand y_d, we consider $n \cdot TL(q_n(r_0)) = n \cdot TL(\frac{1}{2}(n r_0^t + r_0 n^t) - \frac{1}{3}(r_0 \cdot n)\mathbb{I})$. This is a homogeneous quadratic function of n and a homogeneous linear function of r_0. Consequently, it can be written as $y_d \cdot r_0$ where y_d is a vector-valued quadratic function of n.

Given our assumption that $r_0 \neq 0$, the above linear system is under-determined. Therefore, the determinant of the matrix formed by y_f, y_g, and y_d must be zero. This determinant is a quartic polynomial of n, which we refer to as $j(n)$. Consequently, when P_n is a transition plane, we have

$$j(n) = 0 \tag{22}$$

We now return to the characterization of a transition point p_0 for $LT(x,y,z) = T_0 + xT_X + yT_y + zT_z$. Its unit non-repeating eigenvector n must satisfy

$$h(n) = 0 \tag{23}$$
$$j(n) = 0 \tag{24}$$

on $\mathbb{RP}^2$, the two-dimensional real-projective space. To study the maximum number of solutions to the system, we borrow Bézout Theorem from Algebraic Geometry [3]:

Theorem 2. *Let f_0 and g_0 be two homogeneous polynomials in three variables of degree d and e, respectively. Let C_f and C_g be the curves defined by $f_0 = 0$ and $g_0 = 0$ in the complex projective space $\mathbb{CP}^2$. Assume that C_f and C_g do not have any common component, then they intersect at exactly $d * e$ points in $\mathbb{CP}^2$, counted with multiplicity.*

By Bézout's Theorem, there can be at most $8 = 2 \times 4$ solutions as h and j are quadratic and quartic, respectively. This leads to the following result:

Theorem 3. *Under structurally stable conditions, a 3D linear tensor field has at most eight transition points.*

In addition, we have the following result:

Theorem 4. *Under structurally stable conditions, a 3D linear tensor field has an even number of transition points, counting multiplicity.*

A degenerate curve is divided into wedge segments and trisector segments by transition points. Under structurally stable conditions, a degenerate curve extends to infinity on both ends. Consequently, we classify a degenerate curve as either a WW curve (the two end segments are both wedge segments), a WT curve (one end segment is a wedge segment and the other segment is a trisector segment), and a TT curve (the two end segments are both trisector segments).

Along a WW or TT degenerate curve, there must be an even number of transition points. In contrast, along a WT degenerate curve, there must be an odd number of transition points. In particular, the following is true.

Theorem 5. *Under structurally stable conditions, a WW or TT degenerate curve can have at least zero transition points and at most eight transition points. In addition, a WT degenerate curve can have at least one transition point and at most seven transition points.*

This leads to the following result regarding the lower-bound of the number of transition points:

Theorem 6. *Under structurally stable conditions, a 3D linear tensor field can have as few transition points as the number of odd degenerate curves in the field.*

Knowing that there are either two or four degenerate curves in a linear tensor field, we have the following nine scenarios:

1. Two WW curves
2. Two WT curves
3. Two TT curves
4. Four WW curves
5. Two WW curves and two WT curves
6. One WW curve, two WT curves, and one TT curve
7. Four WT curves
8. Two WT curves and two TT curves
9. Four TT curves

Each scenario can be encoded as (p,q,r) where p, q and r are the number of WW curves, WT curves, and TT curves, respectively. For example, the scenario of two WW curves is encoded as $2/0/0$.

Our experiments have shown that both the upper-bound and the lower-bound can be reached for each of the nine scenarios (upper-bound: Figure 2; lower-bound: Figure 3). Therefore, these bounds are not only tight in general, there are tight for each of the nine scenarios. In particular, there can be zero transition points in the field (Figure 3 (a, c, d, and i).

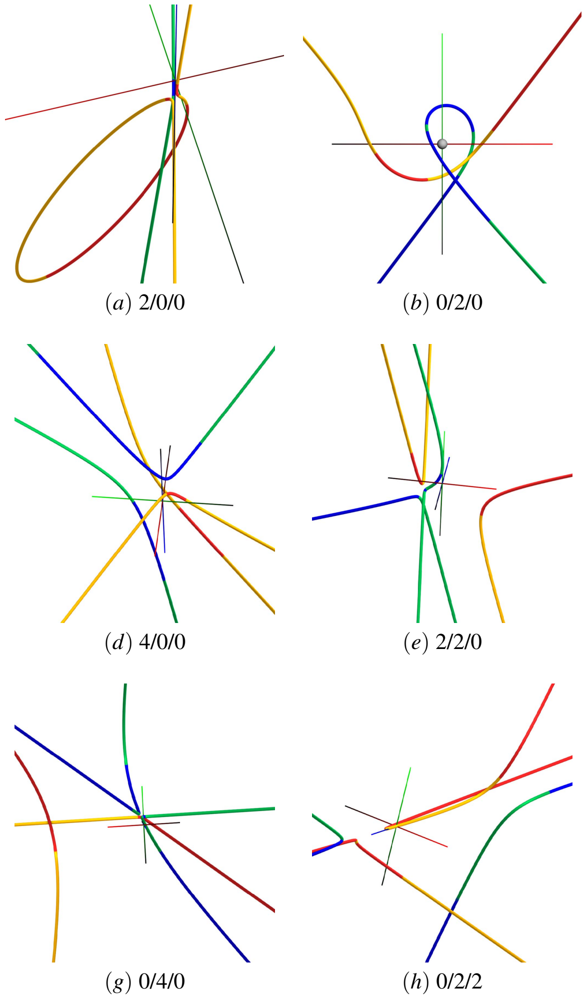
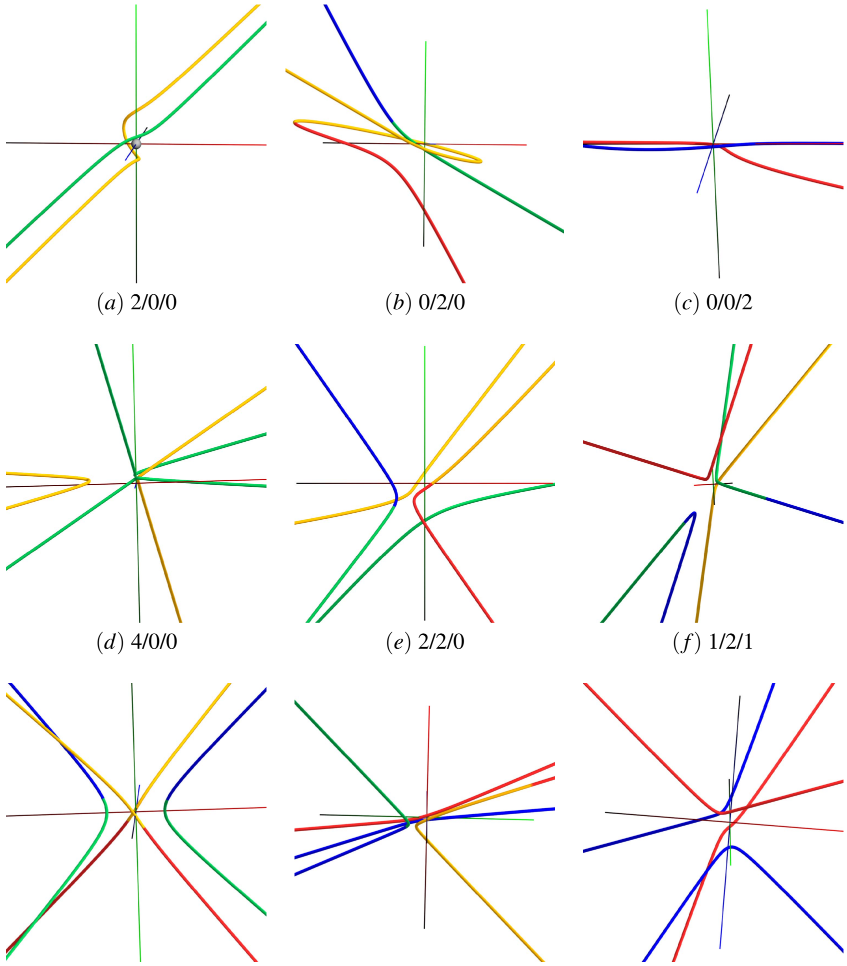


Fig. 2 This figure shows that for each of the nine configurations, it is possible to have eight transition points (theoretical upper bound) in the field. From left to right, the numbers in each triple are the number of WW curves, WT curves, and TT curves, respectively.

5 Conclusion

In this paper, we study the number of transition points in a 3D linear tensor field. We show that under structurally stable conditions there are at most 8 transition points. Moreover, we show that the minimum number of transition points is the same as the number WT degenerate curves in the field. Both of these bounds are tight for the nine scenarios in a linear tensor field.



(*a*) 2/0/0 (*b*) 0/2/0 (*c*) 0/0/2

(*d*) 4/0/0 (*e*) 2/2/0 (*f*) 1/2/1

(*g*) 0/4/0 (*h*) 0/2/2 (*i*) 0/0/4

Fig. 3 This figure shows that for each of the nine configurations, it is possible to have as few transition points as the theoretical lower bound (the number of WT curves) in the field.

In addition, we have established the theoretical lower-bound and upper-bound on the number of transition points that can occur on a single degenerate curve, which are zero (min) and eight (max) for WW and TT curves and one (min) and seven (max) for WT curves.

In practice, we have the lower-bounds for WW, WT, and TT degenerate curves to be tight, i.e., there are tensor fields which have degenerate curves with the given lower-bound transition points. Furthermore, we have also identified 7 to be a tight upper-bound for WT degenerate curves. On the other hand, the observed upper-bound for a WW or TT degenerate curve is *six*, which is a conjecture that we plan to investigate.

In the future, we plan to strive for a tight upper bound on the number of transition points on one WW degenerate curve and on one TT degenerate curve in a 3D linear tensor field. In addition, we plan to study the bifurcations in a 3D tensor field.

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